STEADY NATURAL CONVECTION IN A CIRCULAR, HORIZONTAL PIPE WITH A HARMONICALLY VARIABLE CROSS-SECTION

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1. We shall consider the problem of natural convection in a horizontal pipe whose cross-section varies harmonically along the axis, the variation being small. The tube forms a cavity in an infinite, solid mass, and in it, at infinity, the horizontal gradient of temperature normal to the axis of the tube is prescribed.

Let the axis z be directed along the axis of the tube, the x-axis being directed vertically upwards. The equation of the tube surface is assumed to be

$$x^2 + y^2 = R^2 (1 + \varepsilon f)^2, \qquad \left(f(z) = \sin \omega z, \ \omega = \frac{2\pi}{\lambda}, \ \varepsilon = \frac{a}{R} < 1\right) \tag{1.1}$$

Here R denotes the mean radius of the tube; λ, the spatial period of the wave on the tube; a, the greatest absolute departure of the radius of the tube from R (all succeeding derivations remain valid also in the general case when f is an arbitrary periodic function of z, on condition that it possesses continuous derivatives up to order two).

The equation for natural convection can be written [1]

$$\mathbf{V}\nabla\mathbf{V} = -\frac{1}{\rho_0}\nabla p' + \nu\Delta\mathbf{V} + \beta g\gamma_0 T, \qquad \operatorname{div}\mathbf{V} = 0 \tag{1.2}$$

$$\mathbf{V}\nabla T = \chi\Delta T, \qquad \Delta T_e = 0 \tag{1.3}$$

Here $\mathbf{V} = \{V_x, V_y, V_z\}$ denotes the velocity vector in the fluid, T and T_e, respectively, denote the temperature of the fluid and the solid both measured with respect to a mean temperature T_0, averaged over the cavity; p' is the pressure measured with respect to its equilibrium value P_0 and at $T = T_0$; we introduce the following notation and contractions

$$\eta_0, \ \varkappa, \ \beta_0, \ c_p, \ \nu = \eta_0/\rho_0, \quad \chi = \varkappa/\rho_0 c_p$$

They are, in order, the viscosity, the thermal conductivity, coefficient of thermal expansion of the fluid, specific heat of the fluid at constant pressure, kinematic viscosity and thermal diffusivity of the fluid. We shall assume that all these parameters are independent of temperature. The symbol g denotes the gravitational acceleration and γ_0 is a unit vector in the direction of the acceleration of gravity.

We shall solve Equations (1.2) and (1.3) subject to the following boundary conditions [2]:

$$[V_x]_S = [V_y]_S = [V_z]_S = 0 \tag{1.4}$$

In the solid mass, at a large distance from the cavity, a constant temperature gradient

$$\left[\frac{\partial T_e}{\partial x}\right] = A \qquad \text{at} \quad r \to \infty \tag{1.5}$$

is prescribed.

The heat flux and the temperature are constant across the boundary of the cavity, which gives

$$[T - T_e]_S = 0, \qquad \left[\varkappa \frac{dT}{dn} - \varkappa_e \frac{dT}{dn}\right]_S = 0 \tag{1.6}$$

where κ, κ_e denote, respectively, the thermal conductivities of the fluid and the solid mass, and n denotes a normal to the tube surface.

2. The waviness of the surface of the pipe causes considerable mathematical difficulties when an attempt is made to satisfy the boundary conditions. It turns out that the problem can be solved more easily if transformed coordinates are used. Instead of x, y, z, we shall use

$$\mathrm{x} = \frac{x}{1 + \varepsilon f(z)}, \qquad \mathrm{y} = \frac{y}{1 + \varepsilon f(z)}, \qquad \mathrm{z} = z \tag{2.1}$$

It is easy to see that x, y, z as well as x, y, z are mutually independent, and that in the new system of coordinates, the equation of the surface transforms into that of a circular cylinder of radius R.

Performing the transformation of coordinates (2.1) in Equations (1.2), (1.3) and in the boundary conditions (1.4)-(1.6), we introduce dimensionless quantities, taking the mean radius R as a reference length. In this manner, in the equations as well as in the boundary conditions, it is necessary to perform transformations according to the following formulas:

$$x = l\xi, \qquad y = l\eta, \qquad z = R\zeta \qquad (l = R[1 + \varepsilon f(z)]) \tag{2.2}$$

$$V_x = \frac{\nu}{R} u, \qquad V_y = \frac{\nu}{R} v, \qquad V_z = \frac{\nu}{R} w$$
$$T = AR\theta, \qquad T_e = AR\theta_e, \qquad p' = \frac{\nu^2}{R}\rho_0 p \tag{2.3}$$

Here, u, v, w, θ, θ_e, p are functions of ξ, η, ζ.

In these transformed coordinates, the equation of the surface of the pipe whose cross-section is variable becomes identical with that of the surface of a circular cylinder whose radius is equal to unity. Furthermore, owing to the transformation (2.1), the interior points of the cavity transform into interior points, and those on the boundary, equally, transform into points on the boundary.

If the waviness of the pipe can be assumed to be small, we can show with the aid of simple calculations that it is possible to derive an approximate equation of the problem in the new form, with an accuracy up to terms which appear multiplied by ϵ to the first power, and the same applies to the boundary conditions.

3. The problem considered is "of the type without a threshold" [2,3]. Consequently, in solving it we can use the method of successive approximations, assuming a solution expanded in powers of the Grashoff number γ. In order to solve the resulting linear differential equations, each corresponding to a power of the Grashoff number, we shall in turn use series expansions in terms of the wave amplitude of the tube ϵ, and once again use the method of successive approximations.

We are now looking for solutions in the form of series in powers of γ, i.e.

$$\begin{aligned} u &= \gamma u_1 + \gamma^2 u_2 + \ldots, & p &= \gamma p_1 + \gamma^2 p_2 + \ldots \quad (\gamma = \beta_0 \nu^{-2} g R^4 A) \\ v &= \gamma u_1 + \gamma^2 v_1 + \ldots, & \theta &= \theta_0 + \gamma\theta_1 + \gamma^2\theta_2 + \ldots \\ w &= \gamma w_1 + \gamma^2 w_2 + \ldots, & \theta_e &= \theta_0' + \gamma\theta_1' + \gamma^2\theta_2' + \ldots \end{aligned} \tag{3.1}$$

Substituting the series (3.1) into the equation of the problem, we shall obtain equations for the successive determination of the functions θ_0, θ_0', u_1, v_1, w_1, p_1, θ_1, θ_1', u_2, v_2, w_2, p_2, θ_2, θ_2', ...

Restricting ourselves to the first approximation in terms of γ, we shall in turn seek solutions for these functions, in the form of the linear parts of series in terms of ϵ:

$$\theta_0 = \theta_{00} + \varepsilon\theta_{01}, \qquad \theta_0' = \theta_{00}' + \varepsilon\theta_{01}' \tag{3.2}$$

$$\begin{aligned} u_1 &= u_{10} + \varepsilon u_{11}, & v_1 &= v_{10} + \varepsilon v_{11}, & w_1 &= w_{10} + \varepsilon w_{11} \\ p_1 &= p_{10} + \varepsilon p_{11}, & \theta_1 &= \theta_{10} + \varepsilon\theta_{11}, & \theta_1' &= \theta_{10}' + \varepsilon\theta_{11}' \end{aligned} \tag{3.3}$$

The coefficients of the truncated series (3.2) and (3.3) are functions of ξ, η, ζ. They are determined by linear differential equations of second order, each provided with suitable boundary conditions.

Exact solutions for the functions θ_{00}, θ_{00}', u_{10}, v_{10}, w_{10}, p_{10}, θ_{10}, θ_{10}' are known. The equations for θ_{01}, θ_{01}', u_{11}, v_{11}, w_{11}, p_{11}, θ_{11}, θ_{11}' have been solved by us. In this manner we have determined the zero-order and first-order approximations with respect to both parameters γ and ϵ for the temperature, velocity and pressure functions.

4. In every approximation for temperature and velocity in the expansion in powers of the Grashoff number, the zero-order approximations with respect to the wave amplitude ϵ the equations are identical with the known results for the pipe of constant cross-section [2,3]; the first-order and higher approximations in terms of ϵ represent the effects of waviness on the processes of convection. The first-order approximation for velocity in the expansion in the terms of Grashoff number together with the first approximation in the series in terms of ϵ represents the motion of the fluid and shows that it differs from that in a circular tube by the presence of longitudinal and radial non-zero components, in addition to the aximuthal component. The additional components depend on twice the angle. The variation along the axis of all components of velocity is sinusoidal with a period equal to that of the tube. The first approximation in the expansion in terms of the Grashoff number for temperature together with the first approximation in ϵ depends on three times the angle (in the case of a circular cross-section such type of dependence appears only in the second terms of the series in Grashoff numbers); the temperature varies along the tube harmonically, but is phase angle is displaced by a quantity which vanishes for $\kappa = \kappa_e$.

Higher-order terms in the expansion in terms of ϵ constitute small scale motions; for example, the second term for velocity varies periodically along the tube with a double period compared to that of the tube cross-section.

The results obtained during this investigation can be generalized and used for the study of temperature distributions in horizontal pipes whose cross-sections vary along the axis according to a more complicated law, because at each step the equations are linear and hence the equation of the surface of the tube can be expanded into a Fourier series.

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